Comparing Different Improvement Programs for the N-Vector Model

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Abstract

We discuss the connection between various types of improved actions in the context of the two-dimensional σ-model. We also discuss spectrum-improved actions showing that these actions do not have any improved behaviour. An $O(a^2)$ on-shell improved action with all couplings defined on a plaquette and satisfying reflection positivity is also explicitly constructed.

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In recent years there has been much work in improving lattice actions. The idea behind all these attempts is to modify the lattice action with the addition of irrelevant operators in order to reduce lattice artifacts: in this way one hopes to have scaling (and finite-size scaling) at smaller correlation lengths.

There have been many different approaches to the problem of improving lattice actions. In this letter we will address the problem in the context of two-dimensional N-vector models, trying to point out differences and similarities of the various approaches.

The first systematic study of improvement of lattice actions is due to Symanzik [1]. The idea is very simple. Consider on a square lattice the standard action

$$S^{std} = N \sum_{x\mu} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} \tag{1}$$

where the fields $\boldsymbol{\sigma}_x$ are N-component spins; the partition function is

$$Z = \int \prod_x d\boldsymbol{\sigma}_x \delta(\boldsymbol{\sigma}_x^2 - 1)\, e^{\beta S} \quad . \tag{2}$$

Then consider the one-particle irreducible Green's functions $\Gamma^{(n)}(p_1, \ldots, p_n)$. It is easy to see that at tree level

$$\Gamma^{(n)}(p_1, \ldots, p_n) = \Gamma^{(n)}_{cont}(p_1, \ldots, p_n) + O(a^2) \tag{3}$$

where $\Gamma^{(n)}_{cont}(p_1, \ldots, p_n)$ is the *continuum* n-point function.

The strategy proposed by Symanzik consists in modifying the action so as to cancel the terms of $O(a^2)$ in (3). The simplest action which satisfies this condition at tree-level is the improved action

$$S^{Sym} = N \sum_{x\mu} \left(\frac{4}{3} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} - \frac{1}{12} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+2\mu} \right) \quad . \tag{4}$$

In the Symanzik approach one can proceed further in two different directions: first of all one can improve the action to order $O(a^4)$, $O(a^6)$ and so on. This does not seem to be particularly interesting: indeed, even if the action is tree-level improved to order $O(a^{2k})$, $k > 1$, then corrections of order $O(a^2)$ will again appear at one- and higher-loop order. The second important characteristic of the Symanzik approach is that it can be systematically extended to higher loops: in other words one can remove terms of order $O(g^2 a^2)$, $O(g^4 a^2)$ and so on within perturbation theory.

A second approach to the improvement of lattice actions (the so-called on-shell improvement) has been put forward by Lüscher and Weisz [2]. The idea here is to improve only spectral quantities like the masses of stable particles. In the $O(N)$ σ-model, in a strip $L \times \infty$, one can consider the mass gap $\mu(\beta, L)$ defined by

$$\mu(\beta, L) = - \lim_{y \to +\infty} \frac{1}{y} \log \left[\sum_{x=1}^{L} \langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_{(x,y)} \rangle \right] \quad . \tag{5}$$

Then one considers the asymptotic expansion of $\mu(\beta, L)$ for $\beta \to \infty$ at L fixed which will have generically the form

$$\mu(\beta, L)L \,=\, \sum_{n=1}^{\infty} \frac{\mu_n(L)}{\beta^n} \quad . \tag{6}$$

Each coefficient $\mu_n(L)$ has an expansion in powers of $1/L^2$ (with additional logarithms of L). The $O(a^2)$ improved action is then chosen by requiring, order by order in perturbation theory, that $\mu_n(L)$ does not have $1/L^2$ corrections. It should be noticed that $O(a^2)$ improvement of the one-loop term $\mu_2(L)$ gives the $O(a^2)$ tree-level improved action [2].

Let us now discuss the two different approaches in the soluble case of $N = \infty$. Let us consider the generalized action

$$S \,=\, N \sum_{x,y} J(x-y) \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \quad . \tag{7}$$

We will assume the interaction to be local and parity invariant: if $\widehat{J}(p)$ is the Fourier transform of $J(x)$, we will require that $\widehat{J}(p)$ is a continuous function of p, even under $p \to -p$. We will also require invariance under rotations of $\pi/2$, that is we will assume that $\widehat{J}(p)$ is symmetric under exchange of p_1 and p_2. Redefining β we can normalize the couplings so that

$$\widehat{J}(q) = \widehat{J}(0) - \frac{q^2}{2} + O(q^4) \tag{8}$$

for $q \to 0$. We also introduce the function

$$w(q) = -2(\widehat{J}(q) - \widehat{J}(0)) \tag{9}$$

which behaves as q^2 for $q \to 0$. Finally we will require the theory to have the usual (formal) continuum limit: we will assume that the equation $w(q) = 0$ has only one solution for $-\pi \le q_i \le \pi$, namely $q = 0$. We will need the small-q behaviour of $w(q)$: we will assume in this limit the form

$$w(q) = \, \hat{q}^2 + \alpha_1 \sum_{\mu} \hat{q}_\mu^4 + \alpha_2 \, (\hat{q}^2)^2 + O(q^6) \tag{10}$$

where α_1 and α_2 are arbitrary constants. Here $\hat{q}^2 = \hat{q}_1^2 + \hat{q}_2^2$, $\hat{q}_\mu = 2\sin(q_\mu/2)$. Notice that for (4) we have $w(q) = q^2 + O(q^6)$ i.e. $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$.

Let us now study the improvement program à la Symanzik. Let us consider the two-point function in infinite volume. A trivial calculation gives

$$\langle \boldsymbol{\sigma}_0 \cdot \boldsymbol{\sigma}_x \rangle \;\equiv G_V(x) = \; 1 + \frac{1}{\beta} \int \frac{d^2 p}{(2\pi)^2} \frac{e^{ipx} - 1}{w(p)} + O(e^{-4\pi\beta}) \quad . \tag{11}$$

Notice that only the tree-level term appears, all successive ones being zero. The Fourier transform $\widehat{G}_V(q;\infty)$ for $q \neq 0$ is then given by

$$\widehat{G}_V^{-1}(q;\infty) = \beta w(q) + O(e^{-4\pi\beta}) \tag{12}$$

Improvement à la Symanzik requires then $w(q) = q^2 + O(q^6)$, i.e. $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$. When these two conditions are satisfied the action is improved to all orders in perturbation theory.

Let us now consider the mass-gap $\mu(\beta, L)$ in a strip and its perturbative expansion (6). At one loop it is easy to compute [3, 4]

$$\begin{aligned} \mu(\beta,L)L &= \frac{1}{2\beta} + \frac{1}{\beta^2}\left[\frac{1}{4\pi}\log L + \frac{1}{2}(\overline{F}_{00} + \Lambda_0) \right. \\ &\qquad \left. + \frac{\pi}{144L^2}(1-12\alpha_1) + O(L^{-4})\right] + O(\beta^{-3}) \end{aligned} \tag{13}$$

where

$$\overline{F}_{00} = \frac{1}{2\pi}\left(\gamma_E - \log\pi + \frac{1}{2}\log 2\right) \tag{14}$$

$$\Lambda_0 = \int \frac{d^2p}{(2\pi)^2}\left(\frac{1}{w(p)} - \frac{1}{\hat{p}^2}\right) \tag{15}$$

and $\gamma_E \approx 0.577215665$ is the Euler constant. Perturbative improvement requires the cancellation of the $1/L^2$ term and thus gives the condition $1 - 12\alpha_1 = 0$, i.e. $\alpha_1 = \frac{1}{12}$. Now consider the next order. A simple calculation gives

$$\begin{aligned} \mu_3(L) &= \frac{1}{8\pi^2}\log^2 L + \frac{1}{2\pi}(\overline{F}_{00} + \Lambda_0)\log L + \frac{1}{2}(\overline{F}_{00} + \Lambda_0)^2 \\ &+ \frac{1}{L^2}\left[\frac{1}{144}(1-12\alpha_1)\log L + \frac{\pi}{72}(1-12\alpha_1)(\overline{F}_{00}+\Lambda_0) + \frac{1}{48}(12\alpha_1 + 12\alpha_2 - 1)\right] \end{aligned} \tag{16}$$

Requiring the cancellation of the $1/L^2$ term we get $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$. Thus only Symanzik tree-level improved actions are improved at this level. We can go further and compute $\mu_4(L)$: for $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$ the coefficient of $1/L^2$ is

$$\frac{1}{8}\left(\frac{1}{96\pi} - \Lambda_1\right) \tag{17}$$

where

$$\Lambda_1 = \int \frac{d^2p}{(2\pi)^2}\left[\frac{1}{w(p)^2} - \frac{1}{(\hat{p}^2)^2} + \frac{2}{(\hat{p}^2)^3}\left(\alpha_1\sum_\mu \hat{p}_\mu^4 + \alpha_2(\hat{p}^2)^2\right)\right] \quad . \tag{18}$$

Thus improving the three-loop contribution to $\mu(\beta, L)$ gives

$$\Lambda_1 = \frac{1}{96\pi} \quad . \tag{19}$$

Notice that this condition is global, that is it does not only fix the small q^2 behaviour of $w(q)$ but it depends on the behaviour of $w(q)$ over all the Brillouin zone.

A particular action satisfying (19) is given by

$$S^{Sym2} = N\sum_{x\mu}\left[\left(\frac{4}{3} + 15a\right)\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu} + \left(-\frac{1}{12} - 6a\right)\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+2\mu} + a\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+3\mu}\right] \quad . \tag{20}$$

where $a = 0.00836533968(1)$.

A peculiarity of the large-N limit is the fact that, once (19) is satisfied, all subsequent coefficients $\mu_n(L)$ do not have $1/L^2$ corrections: improvement at three-loops is equivalent to improvement to all orders in perturbation theory: Symanzik actions satisfying (19) are improved to all perturbative orders. Of course this feature will not be true for finite values of N.

In conclusion, within Symanzik approach only the two conditions $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$ are required to improve the action to order $O(a^2)$ and to all orders in perturbation theory. In the on-shell program instead an additional condition (equation (19)) is obtained. Notice that naively one would have expected the opposite result since Symanzik approach is intended to improve all Green's functions while the on-shell program considers only spectral quantities. This result is due to the large N limit [1]. Indeed, one can see that condition (19) appears in the Symanzik approach at order $1/N$. At this order one should consider, beside the two-point function also the four-point function. If $\Gamma_{\alpha\beta\gamma\delta}(p,q,r,s)$ is the one-particle irreducible four-point function, we have

$$\Gamma_{\alpha\beta\gamma\delta}(p,q,r,s) = -\frac{1}{N}\left[\delta_{\alpha\beta}\delta_{\gamma\delta}\Delta(p+q) + \delta_{\alpha\gamma}\delta_{\beta\delta}\Delta(p+r) + \delta_{\alpha\delta}\delta_{\gamma\beta}\Delta(p+s)\right] + O(1/N^2) \tag{21}$$

where

$$\Delta^{-1}(p) = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{[w(k)+m_0^2]\,[w(p+k)+m_0^2]} \tag{22}$$

In the perturbative limit

$$\Delta^{-1}(p) = \frac{1}{\beta w(p)}\left[1 + \frac{1}{2\beta}\int \frac{d^2k}{(2\pi)^2}\frac{w(p)-w(k)-w(p+k)}{w(p+k)w(k)}\right] + O\left(e^{-4\pi\beta}\right) \tag{23}$$

If $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$ we have then [4]

$$\Delta^{-1}(p) = \frac{1}{\beta p^2}\left\{1 + \frac{1}{\beta}\left[\frac{1}{4\pi}\log\frac{p^2}{32} - \Lambda_0 + \frac{p^2}{2}\left(\Lambda_1 - \frac{1}{96\pi}\right)\right] + O(p^4\log p^2)\right\} \tag{24}$$

Thus $O(a^4)$-improvement required the condition (19). In the Symanzik approach (19) is thus a one-loop improvement condition for the four-point function. Notice again the peculiarity of the large N limit: improvement of the one-loop result is enough to improve all orders of perturbation theory.

We have checked that $\alpha_1 = \frac{1}{12}$, $\alpha_2 = 0$ and (19) are sufficient conditions to improve both the scaling and the finite-size-scaling behaviour of the theory. Indeed, consider for instance the ratio (in infinite volume)

$$R(\beta) = \left(\frac{\xi_V^{(2)}(\beta)}{\xi_T^{(2)}(\beta)}\right)^2 \tag{25}$$

[1] We thank Giorgio Parisi who told us that this point was already known to K. Symanzik.

where

$$\xi^{(2)}_{\#}(\beta) = \frac{1}{4}\frac{\sum_x |x|^2 G_{\#}(x)}{\sum_x G_{\#}(x)} \tag{26}$$

and

$$G_T(x) = \langle \sigma^a_0 \sigma^b_0; \sigma^a_x \sigma^b_x \rangle \ . \tag{27}$$

Then an easy calculation gives

$$\begin{aligned} R(\beta) &= 6 - 24 e^{-4\pi(\beta-\Lambda_0)} \left[4\pi(12\alpha_1 + 16\alpha_2 - 1)(\beta - \Lambda_0) \right. \\ & \qquad \left. -(8\alpha_1 + 8\alpha_2 - 1 + 32\pi\Lambda_1) \right] + O(e^{-8\pi\beta}\beta) \ . \end{aligned} \tag{28}$$

Improvement requires then

$$12\alpha_1 + 16\alpha_2 - 1 = 0 \ , \tag{29}$$

$$8\alpha_1 + 8\alpha_2 - 1 + 32\pi\Lambda_1 = 0 \ . \tag{30}$$

The first term clearly vanishes for the Symanzik action for which $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$. The second one however requires the additional condition (19).

Let us now consider finite-size scaling. On a strip of width L, we have computed the corrections to the finite-size-scaling function[2] for $\mu(\beta, L)$. In the limit $L \to \infty$, $\beta \to \infty$ with $\mu(\beta, L)L \equiv x$ fixed we find [3, 4]

$$\frac{\mu(\beta,\infty)}{\mu(\beta,L)} = f_\mu(x)\left(1 + \frac{\Delta_{\mu,1}(x)}{L^2}\log L + \frac{\Delta_{\mu,2}(x)}{L^2}\right) \tag{31}$$

with corrections of order $\log^2 L/L^4$. Cancellation of the terms of order $\log L/L^2$ requires $12\alpha_1 + 16\alpha_2 - 1 = 0$, while cancellation of the terms $1/L^2$ requires $\alpha_1 = \frac{1}{12}$, $\alpha_2 = 0$ and the condition (19). Again (19) is necessary to completely eliminate the corrections of order $1/L^2$.

We want to make an additional remark on the on-shell improvement program. For $N = \infty$ tree-level improvement requires the one-loop contribution to $\mu(\beta, L)$ to have corrections of order $O(L^{-4})$ and thus it requires only $\alpha_1 = \frac{1}{12}$; the second coefficient α_2 is arbitrary to this order. This is a peculiarity of the large-N limit. For finite values of N, using the results of [6, 4] , we have

$$L\mu(\beta, L) = \frac{N-1}{2N\beta}\left[1 + \frac{1}{N\beta}(r_1(L) + (N-2)r_2(L)) + O(\beta^{-2})\right] \tag{32}$$

with

$$r_1(L) = -\int \frac{d^2k}{(2\pi)^2}\frac{1}{w(k)}\left[\frac{1}{4}\sum_\mu \frac{\partial^2 w(k)}{\partial k_\mu^2} - 1\right] - \frac{\pi}{12L^2}(12\alpha_1 + 8\alpha_2 - 1) + O(L^{-4}) \ , \tag{33}$$

$$r_2(L) = \frac{1}{2\pi}\log L + \overline{F}_{00} + \Lambda_0 + \frac{\pi}{72L^2}(1 - 12\alpha_1) + O(L^{-4}) \ . \tag{34}$$

[2]The leading term was already computed in [5].

Clearly the $1/L^2$ corrections cancel only if $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$ except for $N = \infty$. Let is notice again [2] that the one-loop computation fixes the improved action at tree-level[3].

We want now to discuss a third type of actions which we will call classically spectrum-improved actions. The idea which has been put forward by the Bern group [7, 8] consists in changing the action so that to improve the dispertion relations. For instance an action is $O(a^2)$-spectrum improved if the equation $w(iE, p) = 0$ gives the continuum dispersion relation $E^2 - p^2 = 0$ modulo terms of order $O(a^4)$. In practice $w(q)$ must have an expansion with $\alpha_1 = \frac{1}{12}$ and α_2 arbitrary, i.e.

$$w(q) = q^2 + \alpha_2 (q^2)^2 + O(q^6) \quad . \tag{36}$$

An example is the action proposed in [7]:

$$S^{diag} = N \sum_x \left(\frac{2}{3} \sum_\mu \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} + \frac{1}{6} \sum_{\hat{d}} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\hat{d}} \right) \tag{37}$$

where $\hat{d}$ are the two diagonal vectors $(1, \pm 1)$. For this action $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = -\frac{1}{12}$.

We will now show that these actions do not show any improved behaviour. First of all [6] these actions are not on-shell tree-level improved in the sense of Lüscher and Weisz [2]. Indeed tree-level improvement requires $\alpha_2 = 0$ (see equations (33) and (34)). For $N = \infty$ it is also easy to see that these actions do not show any improved behaviour compared to the standard action [3]. Consider for instance the finite-size-scaling behaviour of the ratio $\mu(\beta, \infty)/\mu(\beta, L)$. In table (1) we report the value of the deviations from finite-size-scaling (see equation (31))

$$R(L, x) = \left(\frac{\mu(\beta, \infty)}{\mu(\beta, L)} \right) \frac{1}{f_\mu(x)} - 1 \tag{38}$$

for the various actions for $L\mu(\beta, L) \equiv x = 2$ (this is the value of x for which the deviations are larger). Clearly the action (37) has larger corrections that the standard action. On the other hand, as expected, the Symanzik actions (4) and (20) show much smaller deviations from finite-size scaling, especially (20) which is Symanzik improved to all orders in perturbation theory and for which the corrections to finite-size scaling behave as $\log L/L^4$. Notice that the improvement is working even for $L = 4$–6 in spite of the large spatial extent of the Symanzik actions.

[3]On-shell improvement to order a^{2k}, $k > 1$, was discussed in [6]. It was shown numerically that if one chooses the generalization of (4) improved to order $O(a^{2k})$, then $\mu_2(L)$ has corrections of order $O(L^{-2k-2})$. Assuming this result it is trivial to prove rigorously that for *any* action such that $w(q) = q^2 + O(q^{2k+4})$ (i.e. for any $O(a^{2k})$ tree-level Symanzik-improved action) $\mu_2(L)$ has corrections of order $O(L^{-2k-2})$ (i.e. the action is also $O(a^{2k})$ tree-level on-shell improved in the sense of Lüscher and Weisz [2]). Notice that the opposite result is not true: it is *not necessary* that $w(q) = q^2 + O(q^{2k+4})$ to have $\mu_2(L) = \text{leading} + O(L^{-2k-2})$. For instance consider $w(q)$ such that for $q \to 0$

$$w(q) = q^2 + \beta_1 \sum_\mu q_\mu^8 + \beta_2 (q^2)^2 \sum_\mu q_\mu^4 + \beta_3 (q^2)^4 + O(q^{10}) \quad . \tag{35}$$

L	S^{std}	S^{diag}	S^{Sym}	S^{Sym2}
4	0.022022	0.039225	0.0002840	−0.0008378
6	0.012721	0.021359	0.0007414	−0.0001953
10	0.005758	0.009302	0.0003948	−0.0000246
20	0.001806	0.002825	0.0001141	−0.0000011

Table 1: Values of $R(L;x)$ for $x = 2$ for the various actions we have introduced in the text. Here $N = \infty$.

We want now to show that if $\alpha_2 \neq 0$ an on-shell improved action can be obtained by adding a new four-spin coupling.

Indeed consider the continuum action

$$S = -N \int d^2x \left[\frac{1}{2}(\boldsymbol{\sigma} \cdot \Box \boldsymbol{\sigma}) + \frac{\alpha_2 a^2}{2} (\boldsymbol{\sigma} \cdot \Box^2 \boldsymbol{\sigma} - (\boldsymbol{\sigma} \cdot \Box \boldsymbol{\sigma})^2) + O(a^4) \right] \tag{39}$$

where $\Box = \sum_\mu \partial_\mu^2$. We will now show that, by means of an isospectral transformation, one can get rid of the $O(a^2)$ terms[4]. Setting $\boldsymbol{\sigma} = (\boldsymbol{\pi}, \sqrt{1 - \boldsymbol{\pi}^2})$, the equation of motion can be written as

$$\Box \boldsymbol{\pi} - \boldsymbol{\pi}(\boldsymbol{\sigma} \cdot \Box \boldsymbol{\sigma}) + O(a^2) = 0 \ . \tag{40}$$

Now consider the isospectral change of variable

$$\boldsymbol{\pi}' = \boldsymbol{\pi} + \frac{A a^2}{2} (\Box \boldsymbol{\pi} - \boldsymbol{\pi}(\boldsymbol{\sigma} \cdot \Box \boldsymbol{\sigma})) \ . \tag{41}$$

where A is a free parameter. It is immediate to verify that, setting $A = -\alpha_2$ we can rewrite

$$S = -\frac{N}{2} \int d^2x \, (\boldsymbol{\sigma}' \cdot \Box \boldsymbol{\sigma}') + O(a^4) \tag{42}$$

where $\boldsymbol{\sigma}' = (\boldsymbol{\pi}', \sqrt{1 - (\boldsymbol{\pi}')^2})$. As a final check we compute at one loop $\mu(\beta, L)$ for an action of the form (39). On the lattice we consider

$$S = N \sum_{xy} J(x-y) \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y + \frac{\alpha_3 N}{2} \sum_{xyz} K(x-y) K(x-z) (\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y)(\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_z) \tag{43}$$

where $\hat{K}(q) = q^2 + O(q^4)$ for $q \to 0$ and α_3 is a free parameter. For $a \to 0$ the action (43) has the continuum limit (39) if $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = \alpha_3$. Using the results of [6, 4]

Requiring that $\mu_2(L)$ does not have terms of order $1/L^4$ we get two equations (corresponding to $r_1(L)$ and $r_2(L)$) which will not fix completely the three parameters β_i. Thus at order $O(a^6)$ one can have an on-shell action which is a Symanzik-improved one.

[4] Equivalently one can show that the term proportional to a^2 in (39) vanishes because of the equations of motion (40).

we get the expansion (32) where $r_1(L)$ and $r_2(L)$ are given by

$$
\begin{aligned}
r_1(L) &= -\int \frac{d^2k}{(2\pi)^2}\frac{1}{w(k)}\left[\frac{1}{4}\sum_\mu \frac{\partial^2 w(k)}{\partial k_\mu^2} - \frac{\alpha_3}{2}\sum_\mu\left(\frac{\partial \widehat{K}(k)}{\partial k_\mu}\right)^2 - 2\alpha_3\widehat{K}(k) - 1\right] \\
&\quad -\frac{\pi}{12L^2}(12\alpha_1 + 8\alpha_2 - 8\alpha_3 - 1) + O(L^{-4}) \ , \qquad (44) \\
r_2(L) &= \frac{1}{2\pi}\log L + \overline{F}_{00} + \Lambda_0 + 2\alpha_3\int\frac{d^2k}{(2\pi)^2}\frac{\widehat{K}(k)}{w(k)} \\
&\quad +\frac{\pi}{72L^2}(1 - 12\alpha_1) + O(L^{-4}) \ . \qquad (45)
\end{aligned}
$$

As expected the $1/L^2$ corrections vanish for $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = \alpha_3$. An explicit example with all couplings lying in a plaquette is given by

$$
S^{onshell} = S^{diag} - \frac{N}{24}\sum_x\sum_{i=1}^4\left(\sum_{\mu_i}(\boldsymbol{\sigma}_x\cdot\boldsymbol{\sigma}_{x+\mu_i} - 1)\right)^2 \qquad (46)
$$

where μ_1 runs over the vectors $(1,0)$ and $(0,1)$, μ_2 over $(-1,0)$ and $(0,1)$, μ_3 over $(-1,0)$ and $(0,-1)$ and μ_4 over $(1,0)$ and $(0,-1)$ (the sum over i symmetrizes over the four plaquettes stemming from the point x). This action should be equivalent to S^{Sym} for on-shell quantities. However $S^{onshell}$ enjoys an additional property: it is reflection positive for reflections with respect to the line $x_1 = 0$; it is then possible, with a standard construction, to define a transfer matrix [9, 10]. It is also trivial to prove that this action has the correct continuum limit, in the sense that the ordered configuration is the unique absolute maximum of $S^{onshell}$.

The appearance of a four-spin coupling in $S^{onshell}$ suggests a connection between the on-shell improvement program and the perfect actions [8]. The perfect action is indeed an action which is *tree-level* on-shell improved[5] to all orders in a^2 [11]. Work in this direction is in progress.

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